1

FORCES IN THREE DIMENSIONS

1.1 PARALLELOPIPED OF FORCES

If three forces acting at a point be represented in magnitude and direction by three sides of a parallelopiped passing through a point then their resultant is given in magnitude and direction by the diagonal of the parallelopiped passing through that point.

Thus if \( OA, OB, OC \) be three co-terminous edges of parallelopiped then the resultant of forces represented by \( OA, OB, OC \) shall be represented by the diagonal \( OD \) of the parallelopiped both in magnitude and direction. The converse is also true.

If \( X, Y, Z \) be three forces mutually perpendicular at a point \( O \) then their resultant is given by

\[
R^2 = X^2 + Y^2 + Z^2,
\]

and if \((f, g, h)\) be any point on the line of action of \( R \), its equation is given by

\[
\frac{x-f}{X} = \frac{y-g}{Y} = \frac{z-h}{Z}.
\]

1.2 THEOREMS RELATING TO COUPLES

Couples: A system of forces and not lying in the same straight line but such that \( \vec{F}_1 + \vec{F}_2 = \vec{0} \), is called a couple. The two forces are parallel and equal in magnitude but oppositely directed. The perpendicular distance between two parallel lines of the forces is called the arm of the couple. A couple cannot be
Forces in Three Dimensions

reduced to a single force as the other systems.

A couple is represented by a line perpendicular to the plane of the couple and whose length is proportional to the magnitude of the moment. Thus a couple being a directed quantity, the parallelogram and paralleloiped laws will be valid for couples also.

**Theorem I**: The moment of a couple with respect to any axis perpendicular to its plane is independent of the position of the axis.

**Theorem II**: The moment of a couple is the product of the common magnitude of the forces and the perpendicular distance between them.

**Theorem III**: Two couples which act in the same plane and which have the same moment are equivalent.

**Theorem IV**: Two couples which act in parallel planes and which have the same moment are equivalent.

**Theorem V**: Couples are vectors.

**Theorem VI**: If a rigid body, acted upon by a couple, is given a small rotation $d\theta$ about any axis which is parallel to the axis of the couple, the work done by the couple is the product of the angular displacement and the moment of the couple.

**Theorem VII**: Any system of forces acting upon a rigid body, which is not equivalent to a single couple, can be replaced by two forces which are perpendicular to each other, but whose lines of action, in general, do not intersect.

**Theorem VIII**: A couple can be displaced and rotated in its own plane. The position of the couple has no importance as it can be transferred from one plane to any parallel plane.

1.3 Axis of a Couple

To find the axis of the couple, one of whose force acts through the origin and the other along the line

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n},$$

passing through the point $P(x_1, y_1, z_1)$.

Let $F$ be the magnitude of the force of the couple then components of the force $F$ at $P$ are

$$X = F_l, \; Y = F_m, \; Z = F_n.$$

Similarly, the components of force $-F$ at $O$ are $-X, -Y, -Z$.

Draw perpendicular $PA$ from $P$ on the plane $xy$ and perpendicular $AB$ on $x$-axis. Draw $SBS'$ parallel to $OZ$. Introduce at $B$ two equal forces $Z$ along $BS$ and $BS'$, these forces being in equilibrium among themselves do not alter the equilibrium of the system. Now $Z$ at $A$ and $-Z$ at $B$ form a couple of moment $y_1 Z$ about $x$-axis; and $Z$ at $B$ and $-Z$ at $O$ form a couple of moment $-x_1 Z$ about $y$-axis.
Hence, $Z$ at $P$ and $-Z$ at $O$ give two couples of moment $y_1Z$ about $x$-axis and $-x_1Z$ about $y$-axis.

Similarly, the couple $X$ at $P$ and $-X$ at $O$ is equivalent to two couples of moment $z_1X$ about $y$-axis $-y_1X$ about $z$-axis and the couple $Y$ at $P$ and $-Y$ at $O$ give two couples of moments $x_1Y$ about $z$-axis and $-z_1Y$ about $x$-axis.

Thus the couple $F$ at $P$ and $-F$ at $O$ are equivalent to three couples

$$(y_1Z - z_1Y) = F(y_1n - z_1m) \text{ about } x\text{-axis},$$

$$(z_1X - x_1Z) = F(z_1l - x_1n) \text{ about } y\text{-axis},$$

$$(x_1Y - y_1X) = F(x_1m - y_1l) \text{ about } z\text{-axis},$$

Since, the plane of the couple passes through the origin, its axis is therefore

$$x = \frac{y}{y_1n - z_1m} = \frac{z}{z_1l - x_1n} = \frac{z}{x_1m - y_1l}$$

1.4 TO FIND THE RESULTANT OF A GIVEN SYSTEM OF FORCES ACTING AT GIVEN POINTS OF THE BODY

Let us choose an arbitrary point of the body as origin and three mutually perpendicular axes $OX, OY, OZ$ as axes of reference. Let a system of forces $F_i,$
Forces in Three Dimensions

$F_1, F_2, \ldots, F_n$ be acting at the points $P_1(x_1, y_1, z_1), P_2(x_2, y_2, z_2), \ldots, P_n(x_n, y_n, z_n)$.

The force $F_1$ at $P_1$ may be replaced by a force $F_1$ at $O$ and a couple $(F_1, -F_1)$. Components of the force $F_1$ at $O$ are $X_1, Y_1, Z_1$ along the axes and the components of the couple about the axes are $(y_1Z_1 - z_1Y_1), (z_1X_1 - x_1Z_1)$ and $(x_1Y_1 - y_1X_1)$.

Similarly, for all the other forces of the system.

Thus the system of forces acting on the body are equivalent to three components of the forces

$$X = \sum X_i, \quad Y = \sum Y_i, \quad Z = \sum Z_i,$$

and three component couples

$$L = \sum (y_iZ_1 - z_iY_1), \quad M = \sum (z_iX_1 - x_iZ_1), \quad N = \sum (x_iY_1 - y_iX_1).$$

the summation being from 1 to $n$.

The component forces can be compounded into a single force $R$ such that

$$R^2 = X^2 + Y^2 + Z^2,$$

along the line

$$\frac{x}{X} = \frac{y}{Y} = \frac{z}{Z},$$

and three component couples by a single couple $G$, where

$$G^2 = L^2 + M^2 + N^2,$$

and axis is

$$\frac{x}{L} = \frac{y}{M} = \frac{z}{N}.$$

Thus the system of forces reduces to a single force $R$ and a single couple $G$ at an arbitrary point $O$. Such a combination is called a Dname and $X, Y, Z; L, M, N$ are the six components of the dyname $(R, G)$.

1.5 GENERAL CONDITIONS OF EQUILIBRIUM OF A RIGID BODY

A force $R$ and a couple $G$ together cannot produce equilibrium. Since, the couple $G$ can be replaced by two equal and opposite forces one of which acts through the point $O$ where $R$ meets the plane of the couple. This force and $R$ can be compounded into a single force which passes through $O$ and does not meet the other force of the couple; and hence we cannot have equilibrium.

Hence, there can be equilibrium only when $R$ and $G$ are separately zero. But we know that

$$R^2 = X^2 + Y^2 + Z^2, \quad G^2 = L^2 + M^2 + N^2$$

Hence, for equilibrium we must have:
\[ X = 0, \ Y = 0, \ Z = 0; \]
\[ L = 0, \ M = 0, \ N = 0; \]
\textit{i.e.} the sum of the resolved parts of the system of forces parallel to any three axes of coordinates must separately be zero and also the sums of their moments about three axes must separately be zero.

### 1.6 MOMENT OF A FORCE ABOUT A LINE

Let the given line be

\[
\frac{x - x'}{l'} = \frac{y - y'}{m'} = \frac{z - z'}{n'}
\]

and a force \( F \) acts along the line

\[
\frac{x - f}{l} = \frac{y - g}{m} = \frac{z - h}{n}
\]

Now the components of \( F \) along the axis are \( F_l, F_m, F_n \), and components of the moments about the axes are

\[
F(gu - hm), F(hl - fn), F(fm - gl).
\]

Now the shortest distance between two lines is given by

\[
\frac{|f - x' \quad g - y' \quad h - z'|}{\sqrt{l \quad m \quad n \quad l' \quad m' \quad n'}} \div \sin \theta
\]

where \( \theta \) is the angle between two lines.

The shortest distance between \( F \) and \( z \)-axis is

\[
r = \left| \begin{array}{ccc}
    f & g & h \\
    l & m & n \\
    0 & 0 & 1
\end{array} \right| \div \sin \theta.
\]

\[ \therefore \quad r \sin \theta = fm - gl. \]

Now the moment of \( F \) about \( z \)-axis is

\[
fY - gX = F(fm - gl) = Fr \sin \theta.
\]

We can generalize this result:

\textit{The moment of a force about a line is equal to the product of the force, shortest distance and the sine of the included angle.}

\textbf{Alternatively}

\textit{Resolve the force \( F \) into two components, \( Q \) parallel to the line and \( S \) perpendicular to the line; the product of \( S \) and the shortest distance between line of action of \( S \) and the given line is the moment of the force \( F \) about the required line.}

If the lines are parallel then \( \theta = 0 \), therefore moment is also zero.

If \( AB, CD \) are portions of the line of action of \( F \) and the given line and \( V \) be the volume of the tetrahedron whose opposite edges are \( AB, CD \).
\[ V = \frac{1}{6} AB \cdot CD \cdot r \sin \theta. \]

\[ \text{moment} = \frac{6V}{AB \cdot CD} \cdot F. \]

1.7 CONDITIONS OF EQUILIBRIUM OF A RIGID BODY WITH ONE POINT FIXED

Let us take the fixed point \( O \) as the origin and three perpendicular lines through it as axes of coordinates. Let the external forces acting on the body reduce to component forces \( X, Y, Z \) and components couples \( L, M, N \). Let the force of constraint at \( O \) have component \( X', Y', Z' \), parallel to the axes. Applying results of the article 1.5, we have

\[ X + X' = 0, \quad Y + Y' = 0, \quad Z + Z' = 0 \]  
\[ L = 0, \quad M = 0, \quad N = 0. \]  

...(1)  
...(2)

The equations (1) give only the component reactions at \( O \) in terms of the external forces.

The equation (2) give the conditions of equilibrium, which states that the sum of moments of the external forces about any three perpendicular lines passing through the fixed point \( O \) must be separately zero.

If the external forces all act in one plane passing through \( O \), these conditions reduce to the simpler condition that the sum of the moments of the external forces about \( O \) must vanish.

1.8 CONDITIONS OF EQUILIBRIUM OF A RIGID BODY WITH TWO POINTS FIXED

Let two points \( A \) and \( B \) be the fixed points of the body so that the body can turn about the fixed axis \( AB \). Let us take the fixed line \( AB \) as \( z \)-axis and any point \( O \) on it as origin.
Let \( OA = z', \ OB = z'', \ X', \ Y', \ Z', \ X'', \ Y'', \ Z'' \) be components of the forces of constraints at \( A \) and \( B \) respectively. Let the external forces acting on the body reduce to the dynamoe (\( X, Y, Z, L, M, N \)). The conditions of equilibrium of the forces are:

\[
\begin{align*}
X + X' + X'' &= 0 \quad (1) \\
Y + Y' + Y'' &= 0 \quad (2) \\
Z + Z' + Z'' &= 0 \quad (3) \\
L - Y' z' - Y'' z'' &= 0 \quad (4) \\
M + X' z' + X'' z'' &= 0 \quad (5) \\
N &= 0 \quad (6)
\end{align*}
\]

(1) and (5) give \( X', X'' \), (2) and (4) give \( Y', Y'' \) but there is only one relation between \( Z' \) and \( Z'' \) given by (3) hence their values remain indeterminate.

Finally (6) is the only relation between external forces, so that condition of equilibrium is that the sum of the moments of the external forces about the fixed axis \( AB \) must be zero.

**Examples**

1. **Forces Acting on a Cube**

   **Case (i)** Two equal forces \( R \) act on a cube, whose centre is fixed and whose edge is \( 2a \), along the diagonals of adjacent faces which do not meet; show that the moment of the couple which will keep the cube at rest is either \( Ra \sqrt{3} \) or \( Ra \) according to the direction of the forces.

   \[ \text{Fig. 1.5} \]

   Let the centre of the cube be the origin and the axes parallel to the edges of the cube.

   Let two forces each equal to \( R \) act on the faces parallel to the \( yz \) and \( zx \) planes through the points \((a, 0, 0)\) and \((0, a, 0)\) respectively.

   The resolved parts of these forces parallel to the axes are:

   \[
   \left( 0, \pm \frac{R}{\sqrt{2}}, \pm \frac{R}{\sqrt{2}} \right), \quad \left( \pm \frac{R}{\sqrt{2}}, 0, \pm \frac{R}{\sqrt{2}} \right).
   \]
Therefore the moments of the forces about the axes are:

\[ L = \sum (y_i z_i - z_i y_i) = \pm \frac{aR}{\sqrt{2}}, \]
\[ M = \sum (z_i x_i - x_i z_i) = \pm \frac{aR}{\sqrt{2}}, \]
\[ N = \sum (x_i y_i - y_i x_i) = \pm \frac{aR}{\sqrt{2}} = \pm \sqrt{2} aR \text{ or } 0, \]

according to the direction of the forces.

Therefore, either
\[ G = aR \left( \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right) = \sqrt{3} aR, \]

or
\[ G = aR \left( \frac{1}{2} + \frac{1}{2} \right)^{1/2} = aR. \]

**Case (ii)** Six forces each equal to \( P \), act along the edges of the cube, taken in order which do not meet a given diagonal. Show that their resultant is a couple of moment
\[ 2\sqrt{3} Pa, \]
where \( a \) is the edge of the cube.

Let \( OA, OB, OC \), be taken as the axes of coordinates and let the six forces which do not meet the diagonal \( OO' \) be taken as shown in the figure. Then the resolved parts of the forces along the axes are:

\[ X = -P + P = 0, \]
\[ Y = P - P = 0, \]
\[ Z = P - P = 0. \]

The moments of the couples about the axes are:
\[ L = -Pa - Pa = -2Pa, \]
\[ M = -Pa - Pa = -2Pa, \]
\[ N = -Pa - Pa = -2Pa. \]
\[ \therefore \quad G = 2\sqrt{3} Pa. \]
Therefore, the resultant of the forces is a couple of moment \( 2\sqrt{3} Pa \).

**Case (iii)** \( OA, OB, OC \) are the edges of a cube of side \( a \) and \( OO', AA' BB', CC' \) are its diagonals; along \( OB', O'A, BC, C'A' \) act forces equal to \( P, 2P, 3P, 4P \); show that they are equivalent to force \( \sqrt{35} P \) at \( O \) along a line whose direction cosines are proportional to \(-3, -5, 6\) together with a couple \( \frac{Pa}{2} \) about a line whose direction cosines are proportional to \( 7, -2, 2\).

The forces are shown in the figure.

![Fig. 1.7](image-url)

The resolved parts of the forces parallel to the axes are:

\[
\left( \frac{P}{\sqrt{2}}, 0, \frac{P}{\sqrt{2}} \right) \quad \text{at the point (0, 0, 0),}
\]
\[
\left( 0, -\frac{2P}{\sqrt{2}}, -\frac{2P}{\sqrt{2}} \right) \quad \text{at the point (a, 0, 0),}
\]
\[
\left( 0, -\frac{3P}{\sqrt{2}}, \frac{3P}{\sqrt{2}} \right) \quad \text{at the point (0, a, 0),}
\]
\[
\left( -\frac{4P}{\sqrt{2}}, 0, \frac{4P}{\sqrt{2}} \right) \quad \text{at the point (a, a, 0).}
\]
\[ \therefore \quad X = \frac{P}{\sqrt{2}} - \frac{4P}{\sqrt{2}} = -\frac{3P}{\sqrt{2}}, \]
\[ Y = -\frac{2P}{\sqrt{2}} - \frac{3P}{\sqrt{2}} = -\frac{5P}{\sqrt{2}}, \]
Forces in Three Dimensions

\[ Z = \frac{P}{\sqrt{2}} - \frac{2P}{\sqrt{2}} + \frac{3P}{\sqrt{2}} + \frac{4P}{\sqrt{2}} = \frac{6P}{\sqrt{2}}. \]

Therefore the resultant force is

\[ R = \sqrt{X^2 + Y^2 + Z^2} = \sqrt{35}P. \]

and the direction of the line of action of the resultant are proportional to \( X, Y, Z, \text{i.e.,} -3, -5, 6. \)

Components of couple of moments are:

\[ L = \frac{3Pa + 4Pa}{\sqrt{2}} = \frac{7Pa}{\sqrt{2}}, \]

\[ M = \frac{2Pa - 4Pa}{\sqrt{2}} = \frac{-2Pa}{\sqrt{2}}, \]

\[ N = \frac{-2Pa + 4Pa}{\sqrt{2}} = \frac{2Pa}{\sqrt{2}}. \]

Therefore the resultant couple is

\[ G = (L^2 + M^2 + N^2)^{1/2} = \frac{Pa\sqrt{114}}{2}, \]

about a line whose direction cosines are proportional to \( L, M, N, \text{i.e.,} 7, -2, 2. \)

2. Four forces act along generators of the same system of a hyperboloid. Their magnitudes are such that if they were transferred parallel to themselves to act at one point they would be in equilibrium; show that they are in equilibrium when acting along generators.

Let \( P_r (r = 1, 2, 3, 4) \) be the forces acting along four generators of the same system of the hyperboloid

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \]

whose equations are:

\[ \frac{x - a \cos \theta_r}{a \sin \theta_r} = \frac{y - b \sin \theta_r}{-b \cos \theta_r} = \frac{z}{c}. \]

The direction cosines of these lines are:

\[ \frac{a \sin \theta_r}{k_r}, \frac{-b \cos \theta_r}{k_r}, \frac{c}{k_r}, \text{where } k_r^2 = a^2 \sin^2 \theta_r + b^2 \cos^2 \theta_r + c^2. \]

Since, we are given that when the forces are transferred parallel to themselves to act at one point they are in equilibrium, we have

\[ X = \sum \frac{P_r a \sin \theta_r}{k_r} = 0, \]
\[ Y = \sum - \frac{P_r b \cos \theta_r}{k_r} = 0, \]

and

\[ Z = \sum \frac{P_r c}{k_r} = 0. \]

Now

\[ L = \sum (y_i z_i - z_i y_i) = \sum b \sin \theta_r \cdot \frac{P_r c}{k_r} = \frac{b c}{a} \cdot X = 0, \]

\[ M = \sum (z_i x_i - x_i z_i) = \sum (-a \cos \theta_r) \cdot \frac{P_r c}{k_r} = \frac{c a}{b} \cdot Y = 0, \]

\[ N = \sum (x_i y_i - y_i x_i) = \sum a \cos \theta_r \left( \frac{-P_r b \cos \theta_r}{k_r} \right) - b \sin \theta_r \left( \frac{P_r a \sin \theta_r}{k_r} \right) \]

\[ = \sum - \frac{ab P_r}{k_r} = - \frac{ab}{c} \cdot Z = 0. \]

Hence, the forces are in equilibrium.

3. A square table stands on four legs placed respectively at the middle points of its sides; find the greatest weight that can be put at one of its corner without upsetting the table.

Let \(ABCD\) be the top of the table and \(E, F, G, H\), the middle points of its sides.

Suppose the weight is placed at the corner \(C\). If the table overturns about the line joining the feet of the legs at \(E\) and \(F\), and the legs at \(G\) and \(H\) will lose the contact with the ground.

Let \(W\) be the weight of the table acting at its \(C\). \(G, O\) and \(W'\) be the weight placed at \(C\). The distance of \(C\) from \(EF = \) distance of \(O\) from \(EF\).

The table will be just on point of turning if \(W\) and \(W'\) have equal moments about \(EF\).

Hence, Moments of \(W\) and \(W'\) are equal if \(W = W'\)
Therefore the greatest weight that can be put at one corner without upsetting the table is equal to the weight of the table.

4. A round table stands upon three equidistant weightless legs at its edge, and a man sits upon its edge opposite a leg. It just upsets and falls upon its edge and two legs. He then sits upon its highest point and just tips it again.

Show that the radius of the table is, \( \sqrt{2} \) times the length of the leg.

Let \( ABC \) be the top of the table, with legs at \( A, B, C \) and the man sitting at \( P \). The length of the perpendiculars from the centre upon \( BC \) is \( a \cos 60^\circ, \) i.e., \( \frac{1}{2}a \), where \( a \) is the radius of the circle. Hence, the lengths of the perpendiculars both from the centre \( O \) and \( P \) on \( BC \) are equal.

Considering, moments about \( BC \), we see that the weight at \( P \) should be equal to the weight of the table, when it just upsets.

The second figure shows the table in upset position, with point \( P \) on the ground.

Let \( l \) be the length of the leg of the table and \( \angle APK = \theta \), we have

\[
\tan \theta = \frac{ML}{PM} = \frac{l}{a/2} = \frac{2l}{a},
\]

since, the perpendicular from \( P \) on \( BC \) is \( a/2 \).

Now

\[
LP = PM \sec \theta = \frac{1}{2} a \sec \theta \\
KP = AP \cos \theta = 2a \cos \theta.
\]

\[
KL = 2 a \cos \theta - \frac{a}{2} \sec \theta
\]

\[
NL = \frac{1}{2} a \sec \theta - a \cos \theta.
\]

The table will return to its original position, with the man going over to \( A \), if the moment of his weight at \( A \) about \( L \) is just greater than moment of the weight of the table \( L \).

Hence, we have

\[
wNL = wKL
\]
\[
\frac{1}{2}a \sec \theta - \cos \theta = 2a \cos \theta - \frac{1}{2}a \sec \theta,
\]
\[
\therefore \quad \sec \theta = 3 \cos \theta.
\]
\[
\therefore \quad \cos \theta = \frac{1}{\sqrt{3}}, \quad \tan \theta = \sqrt{2}.
\]  
(2)

From (1) and (2) we have \(a = l \sqrt{2}\).

5. A door of weight \(W\), is free to turn about an axis \(AB\) which is inclined at an angle \(\alpha\) to the vertical; show that the couple necessary to keep it in a position in which it is inclined at an angle \(\beta\) to the vertical plane through \(AB\) is \(Wa \sin \alpha \sin \beta\), where \(a\) is the distance of its centre of gravity from \(AB\).

\[
AB \text{ is the axis of the door inclined at an angle } \alpha \text{ to the vertical and } LAB \text{ is the vertical plane through } AB. \text{ When the door is held in the position } ABCD, \text{ let } AD \text{ makes an angle } \beta \text{ with } AL. 
\]

The weight \(W\) of the door acting through its centre of gravity \(G\) has components \(W \cos \alpha\) parallel to \(AB\) and \(W \sin \alpha\) parallel to \(AL\).

Let \(AM\) be the perpendicular to the plane \(ABCD\). Let \(AD, AM, AB\), be the axes of coordinates. Hence components of \(W\) are
\[
X = W \sin \alpha \cos \beta, \quad WY = W \sin \alpha \sin \beta, \quad Z = W \cos \alpha
\]

The moment of \(W\) about \(AB\) is \(Y \cdot a\), i.e., \(Wa \sin \alpha \sin \beta\), which is also the moment of the couple necessary to keep the door in the position \(ABCD\).

6. A rectangular gate is hung is the ordinary way on two hinges so that the line joining the hinges makes an angle \(\alpha\) with the vertical. Show that the work which must be done to move it through an angle \(\theta\) from its position of equilibrium is \(Wa \sin \alpha (1 - \cos \theta)\), where \(W\) is the weight and \(2a\) is the breadth of the gate.
From previous example, we see that work done in moving the gate through an angle $\delta \beta$ from the position $ABCD$ is

$$Wa \sin \alpha \sin \beta \cdot \delta \beta.$$ 

Therefore the work done in moving the gate through an angle $\theta$ from its position of equilibrium (in the plane $BAL$)

$$= \int_0^\theta Wa \sin \alpha \sin \beta \cdot \delta \beta \cdot \delta \beta = Wa \sin \alpha (1 - \cos \theta)$$

7. A uniform straight rod, of length $2c$, is placed in a horizontal position as high as possible within a hollow sphere, of radius $a$. Show that the line joining the middle point of the rod to the centre of the sphere makes with the vertical an angle $\tan^{-1} \left( \frac{\mu \alpha}{\sqrt{\alpha^2 - c^2}} \right).$

Let $AB$ be the rod resting horizontally inside a rough sphere such that the line joining its middle point $C$ to the centre $O$ of the sphere makes the greatest angle $\theta$ with the vertical.

Since, the rod tends to slip down always remaining horizontal, the point $C$ will describe a circle of radius $OC$ about $O$ in the vertical plane.

The reactions at $A$ and $B$ are each equal to $R$ acting along the radii $AO$ and $BO$, making the same angle $\alpha$ with $AB$. Hence, their resultant is $2R \sin \alpha$ acting along $CO$, where $\cos \alpha = \frac{c}{a}$.

The frictional forces at $A$ and $B$ are each $\mu R$ at right angles to $OC$ in a vertical plane. Hence, their resultant is $2\mu R$ acting at $C$ at right angles to $CO$, making an angle $\theta$ with the horizontal.

For equilibrium of the rod, we have

$$2R \sin \alpha \cos \theta + 2\mu R \sin \theta = W,$$

$$2R \sin \alpha \sin \theta = 2\mu R \cos \theta.$$
8. A heavy circular cylinder rests with plane base upon a rough horizontal table; if its weight be $W$ and the normal pressure be supposed to be uniformly distributed over the base, show that the moment of the couple about its axis which would just twist it is \( \frac{2}{3} \mu Wa \), where \( \mu \) is the coefficient of friction and \( a \) is the radius of the base.

It is evident that total normal reaction on the base of the cylinder is equal to the weight of the cylinder. Since this reaction is uniformly distributed over the whole base, the reactions per unit area of the base is \( \frac{W}{\pi a^2} \).

Let us consider an elementary area \( r \, \theta \, dr \) of the base. The reaction on this area is \( \left( \frac{W}{\pi a^2} \right) r \, \theta \, dr \). The moment of the friction about the axis of the cylinder is \( \left( \frac{\mu W}{\pi a^2} \right) r \, \theta \, dr \).

Therefore the frictional moment on the entire base, which is equal to the couple required to turn the cylinder

\[
\frac{\mu W}{\pi a^2} \int_{r=0}^{r=a} \int_{\theta=0}^{\theta=2\pi} r^2 \, d\theta \, dr = \frac{\mu W}{\pi a^2} \frac{1}{3} a^3 \cdot 2\pi = \frac{2}{3} \mu Wa .
\]

9. A right circular cone, of weight $W$ and vertical angle $2\alpha$, is placed with its vertex downwards and supported by a circular hole cut in a horizontal table. If \( \mu \) be the coefficient of friction and \( b \) the radius of the hole, show that the moment of the least couple that will move the cone is $\mu W b \csc \alpha$. 

The second equation gives

\[
\tan \theta = \frac{\mu}{\sin \alpha} = \frac{\mu a}{\sqrt{a^2 - c^2}} .
\]
Let us consider an element $bd\theta$ situated at the point $P$ of the circular hole of the radius $b$ into which the cone rests.

If $R$ be the reaction per unit length of the arc of the hole, the reaction of this element is $Rbd\theta$, acting at right angles with generator $OP$ of the cone through $P$.

The components of this reaction are $Rbd\theta \cos \alpha$ in the plane of the hole passing through its centre $C$ and $Rbd\theta \sin \alpha$ vertical.

It is evident that the horizontal reaction of the different parts of the hole balance each other, while vertical reactions support the weight of the cone. Therefore,

$$W = \int_{0}^{\theta=2\pi} Rbd\theta \sin \alpha = 2\pi Rb \sin \alpha.$$

The frictional force of the element at $P$ is $\mu Rbd\theta$ along the tangent to the arc, and its moment about the axis $OA$ of the cone is $\mu Rbd\theta b$. Therefore total moment.

$$= \int_{0}^{2\pi} \mu Rb^2 d\theta = 2\pi \mu Rb^2 = \mu bW \csc \alpha.$$

### 1.9 VIRTUAL WORK

Suppose that a body is in equilibrium under the action of any system of forces. Let the body be imagined to undergo a slight displacement consistent with the geometrical conditions, such that the point $A$ of the body is displaced to the point $A'$, then $AA'$ is called virtual displacement and the work done during this displacement is called virtual work. The word virtual implies that this displacement is not actual, it is only hypothetical or imaginary. There is nothing essentially different from real work.

If $AA'$ makes an angle $\theta$ with the line action of the force $\vec{F}$ at $A$ then work done is $\vec{F}.\overrightarrow{AA'}$. Like actual work virtual work is also a vector quantity. Although