What is motion? The phrase “change of position with time” expresses the basic idea of motion. We say that an object is moving if it occupies different positions in space at different times. In contrast, if an object does not change its position in space as time flows, we say that the object is at rest. Inherent in this description of rest and motion is the observer (that is ‘we’) and the concepts of a separate space and time.

Our perception tells us that space exists as a stationary, permanent and absolute background in which we can place an object, or through which an object can move without any interaction with it. Newton formulated this idea by asserting, “Absolute space, in its own nature, without relation to anything external, remains always similar and immovable”.

Time, according to Newton, is also absolute and flows on without relation to the presence of any physical object or event. In his words, “Absolute, true, and mathematical time, of itself, and from its own nature, flows equally without relation to anything external, and by another name is called duration.” In this Newtonian absolute space and time, motion of an object is characterized by change of its position in space as time evolves.

However, since this abstract absolute space cannot be seen and/or no part of it can be distinguished from another, how do we fix the position of an object in this space? Newton regarded the distant stars in the sky at rest and defined the positions of other objects in relation to them. However, astronomers now tell us that none of these cosmological objects in the Universe are at rest. There is no object at rest in this ‘absolute space’ with respect to which we can define the position of another object. Therefore, at any instant of time, the position of an object is always defined by ‘the observer’ with respect to himself. An observer, therefore, first defines a local space, in which positions of other objects are then described. This local space, to which is attached the observer, is called a frame of reference.

The concepts of rest and motion are therefore relative, that is, these are in relation to a frame of reference or observer. A house built on Earth does not change its position in the local space or frame of reference associated with Earth; however, as observed from a frame of reference attached to Moon for example (or in other words, in the local space associated with Moon), its (house on Earth) position continuously changes. Thus, even within the Newtonian point of view, space does not exist on its own but is a relative space. Time, on
the other hand, is same and flows equably everywhere, i.e., for example its rate of flow is same both on Earth and Moon.

1.1 FRAMES OF REFERENCE AND CO-ORDINATE SYSTEMS

A frame of reference is usually a collection of objects, together with the observer, which are at rest relative to each other*. Thus, for example, we at rest relative to Earth along with Earth and laboratories built on Earth, etc., constitute a frame of reference. It is with respect to this Earth’s frame of reference that we observe and measure the changes of positions of other objects. Similar frames of reference can be associated with other observers (or objects) who may not be at rest relative to us.

Within a frame of reference, we set up a co-ordinate system which is used to measure the position of an object. Measurements indicate that the space we inhabit is a three-dimensional space. This means that we need three numbers to specify the position of any (point) object in the space. The choice of numbers is determined by the type of co-ordinate system that we use. The generally used co-ordinate systems are rectangular (Cartesian) co-ordinates \( (x, y, z) \), spherical-polar co-ordinates \( (r, \theta, \phi) \), and cylindrical co-ordinates \( (\rho, \phi, z) \).

A co-ordinate system may be set up anywhere in the frame of reference, the choice of origin has complete freedom. Given the origin, a Cartesian (rectangular) system is formed by having three mutually perpendicular straight lines, the \( X \), \( Y \), and \( Z \) axes. There are two independent rectangular systems possible in space, the left-handed and the right-handed, which cannot be made to coincide with each other by any means of translation or rotation. Cartesian system, by convention, is chosen to be right-handed, as shown in Fig. 1.1. The position of a particle** \( P \) relative to origin \( O \) is given by position vector \( \mathbf{r} \), characterized by its specific length and direction:

\[
\mathbf{r} = xi + yj + zk \tag{1.1}
\]

where \( i, j, k \) denote constant unit vectors in the directions of \( X, Y, Z \) axes.

We talk of a point, straight line, perpendicular straight lines, length of line segment, etc. in the Euclidean sense. Our experience tells us that Euclidean geometry is applicable in our ordinary Newtonian space. This is valid if the objects are not very massive, or are not moving with very high speeds of the order of speed of light.

If the position of the particle \( P \) changes, the differential change in its position vector \( \mathbf{r} \) is, in general, given by,

\[
d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} \tag{1.2}
\]

---

* The collection of such point objects constitute the concept of a rigid body.

** We define particle as a mass concentrated at a point. The particle is therefore a point object.
The instantaneous velocity and acceleration of $P$, in Cartesian system, therefore are

$$ v = \frac{dr}{dt} = \frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k $$

$$ = v_x i + v_y j + v_z k $$

...(1.3)

and

$$ a = \frac{dv}{dt} = \frac{dv_x}{dt} i + \frac{dv_y}{dt} j + \frac{dv_z}{dt} k $$

$$ = a_x i + a_y j + a_z k $$

...(1.4)

### 1.1.1 Spherical-Polar Co-ordinates

The position of particle $P$ relative to origin $O$, in spherical polar co-ordinate system, is described by radial distance $r$, polar angle $\theta$, and azimuth $\phi$, as shown in Fig. 1.2a. Geometry of the figure shows the relations between Cartesian and spherical polar co-ordinates:

$$ x = r \sin \theta \cos \phi $$

$$ r = \sqrt{x^2 + y^2 + z^2} $$

...(1.5a)

$$ y = r \sin \theta \sin \phi $$

$$ \tan \theta = \frac{\sqrt{x^2 + y^2}}{z} $$

...(1.5b)

$$ z = r \cos \theta $$

$$ \tan \phi = \frac{y}{x} $$

...(1.5c)

![Fig. 1.2: Spherical co-ordinates.](image-url)
by the rotation of $OP$ about $Z$-axis; displacement along this circle changes only co-ordinate $\phi$, distance $OP = r$ and angle $\theta$ remain fixed; see Fig. 1.2 (b).

The general differential displacement of particle $P$ in spherical polar co-ordinates is given by,

$$dr = dre_r + r d\theta e_\theta + rsin\theta d\phi e_\phi$$  \hspace{1cm} (1.6)

One can satisfy himself that the above relation is true by considering special cases: (i) if $d\theta = d\phi = 0$, $dr = dr e_r$ gives displacement vector when distance $r$ changes by $dr$ along line OP; similarly, (ii) if $dr = d\phi = 0$, and angle $\theta$ changes by $d\theta$, displacement vector $dr = (r \ d\theta \ e_\theta)$ and (iii) if $dr = d\theta = 0$, change of angle $\phi$ by $d\phi$ produces a displacement $dr = (\rho \ d\phi \ e_\phi)$, where $\rho = rsin\theta$ is the radius of the circle as we rotate $P$ about $Z$-axis.

The unit vectors $e_r, e_\theta$ and $e_\phi$ can be expressed in terms of $i, j, k$ as follows (see Example 1.1 on p.6):

$$e_r = \sin\theta \cos\phi \ i + \sin\theta \sin\phi \ j + \cos\theta \ k$$  \hspace{1cm} (1.7a)

$$e_\theta = \cos\theta \cos\phi \ i + \cos\theta \sin\phi \ j - \sin\theta \ k$$  \hspace{1cm} (1.7b)

$$e_\phi = -\sin\phi \ i + \cos\phi \ j$$  \hspace{1cm} (1.7c)

The unit vectors ($e_r, e_\theta, e_\phi$), unlike ($i, j, k$), are not constant vectors but change in direction as co-ordinates $\theta$ and $\phi$ change. However, at each point, they constitute an orthogonal right-handed co-ordinate system, that is, we have

$$e_r \cdot e_r = e_\theta \cdot e_\theta = e_\phi \cdot e_\phi = 0$$  \hspace{1cm} (1.8a)

$$e_r \times e_\theta = e_\phi, \ e_\theta \times e_\phi = e_r, \ e_\phi \times e_r = e_\theta$$  \hspace{1cm} (1.8b)

1.1.2 Motion in 2-Dimensions

Quite often, we find a particle constrained to move in a plane, rather than in space in general. If we call this plane as $XY$ plane, then position vector of the particle is given by,

$$r = x \ i + y \ j$$

because $z = 0$ always for the particle during motion. The spherical polar co-ordinates $(r, \theta, \phi)$ then reduce to circular polar co-ordinates $(r, \phi)$; $\theta = 90^\circ$ for the particle. That is, we define the position of the particle in terms of radial distance $r$ from origin and angle $\phi$ such that,

$$r = \sqrt{x^2 + y^2} \hspace{1cm} x = r \cos\phi$$  \hspace{1cm} (1.9a)

$$\tan\phi = y/x \hspace{1cm} y = r \sin\phi$$  \hspace{1cm} (1.9b)

The displacement vector $dr$ in circular polar co-ordinate is obtained by putting $\theta = 90^\circ$ and $d\theta = 0$ in Eq. 1.6:
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\[ \dot{\mathbf{r}} = d\mathbf{r} + r d\phi \mathbf{e}_\phi \]  
...(1.10)

where unit vectors \( \mathbf{e}_r, \mathbf{e}_\phi \) are given by,

\[ \mathbf{e}_r = \cos \phi \mathbf{i} + \sin \phi \mathbf{j} \]  
...(1.11a)

\[ \mathbf{e}_\phi = -\sin \phi \mathbf{i} + \cos \phi \mathbf{j} \]  
...(1.11b)

1.1.3 Velocity and Acceleration in Circular Polar Co-ordinates

The velocity and acceleration of a point object, moving in a plane, can be obtained in terms of polar co-ordinates \((r, \phi)\) by differentiating position vector \( \mathbf{r} = r \mathbf{e}_r \) with respect to time:

\[ \mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt} \mathbf{e}_r + r \frac{d\mathbf{e}_r}{dt} \]  
...(1.12)

Note that \( \mathbf{e}_r \) is not a constant vector; its time rate can be obtained by using Eqs. 1.11a, b:

\[ \frac{d\mathbf{e}_r}{dt} = -\sin \phi \frac{d\phi}{dt} \mathbf{i} + \cos \phi \frac{d\phi}{dt} \mathbf{j} = \frac{d\phi}{dt} \mathbf{e}_\phi \]  
...(1.13a)

Similarly, we obtain

\[ \frac{d\mathbf{e}_\phi}{dt} = -\frac{d\phi}{dt} \mathbf{e}_r \]  
...(1.13b)

The velocity vector, therefore, is written as

\[ \mathbf{v} = v_r \mathbf{e}_r + v_\phi \mathbf{e}_\phi \]  
...(1.14)

where \( v_r = \frac{dr}{dt} = \dot{r} \), and \( v_\phi = r \frac{d\phi}{dt} = \dot{\phi} \) are the radial and tangential (or azimuthal) components of velocity vector \( \mathbf{v} \).

To find the expression for acceleration, we have

\[ \mathbf{a} = \frac{d\mathbf{v}}{dt} = \left( \frac{dv_r}{dt} \mathbf{e}_r + v_r \frac{d\mathbf{e}_r}{dt} \right) + \left( \frac{dv_\phi}{dt} \mathbf{e}_\phi + v_\phi \frac{d\mathbf{e}_\phi}{dt} \right) \]

\[ = \left( \frac{dv_r}{dt} - v_\phi \frac{d\phi}{dt} \right) \mathbf{e}_r + \left( v_r \frac{d\phi}{dt} + \frac{dv_\phi}{dt} \right) \mathbf{e}_\phi \]

or

\[ \mathbf{a} = a_r \mathbf{e}_r + a_\phi \mathbf{e}_\phi \]  
...(1.15)

where

\[ a_r = \ddot{r} - \dot{\phi}^2 \]  
...(1.15a)

and

\[ a_\phi = 2 \dot{r} \dot{\phi} + \ddot{\phi} \]  
...(1.15b)

are the radial and tangential components of acceleration vector \( \mathbf{a} \).
In particular, if the object is moving in the plane, in a circle of radius \( R \) (\( r = R \), constant), we find

\[ v_r = 0, \quad v_\phi = R \frac{d\theta}{dt} \]  

...(1.16a)

and

\[ a_r = R \frac{d^2\theta}{dt^2}, \quad a_\phi = R \frac{d\phi}{dt} \]  

...(1.16b)

\( a_r \) denotes the centripetal acceleration (towards \(-e_r\)), and \( a_\phi \) is tangential acceleration responsible for increase in tangential speed \( v_\phi \).

**Example 1.1:** Derive Eqs. 1.7 a, b, c.

**Solution:** We have,

\[ \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = r\sin\theta\cos\phi\mathbf{i} + r\sin\theta\sin\phi\mathbf{j} + r\cos\theta \mathbf{k} \]

Hence,

\[ \frac{\partial \mathbf{r}}{\partial r} = \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta \mathbf{k} \]

\[ \frac{\partial \mathbf{r}}{\partial \theta} = r(\cos\theta\cos\phi\mathbf{i} + \cos\theta\sin\phi\mathbf{j} - \sin\theta \mathbf{k}) \]

and

\[ \frac{\partial \mathbf{r}}{\partial \phi} = r\sin\theta(-\sin\phi\mathbf{i} + \cos\phi\mathbf{j}) \]

Also, from Eq. 1.6, we find

\[ \frac{\partial \mathbf{r}}{\partial \theta} = \mathbf{e}_r (\theta, \phi \text{ const.}) \]

\[ \frac{\partial \mathbf{r}}{\partial \phi} = r \mathbf{e}_\phi (r, \phi \text{ const.}) \]

and

\[ \frac{\partial \mathbf{r}}{\partial \phi} = r\sin\theta \mathbf{e}_\phi (r, \theta \text{ const.}) \]

Comparing above two sets of equations, we get Eqs. 1.7a, b, c.

**1.1.4 Cylindrical Co-ordinates**

In cylindrical co-ordinates, the position of particle \( P \) relative to origin \( O \) is described by the two circular polar co-ordinates \( (\rho, \phi) \) defined in the plane \( Z = 0 \), and the third being the \( Z \)-co-ordinate itself. The cylindrical co-ordinates \( (\rho, \phi, z) \) are related to Cartesian co-ordinates as,

\[ \rho = \sqrt{x^2 + y^2}, \quad \tan\phi = \frac{y}{x}, \quad z = z \]  

...(1.17a)

Conversely,

\[ x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z \]  

...(1.17b)

The equation \( \rho = \text{ constant} \) describes a right circular cylinder of radius \( \rho \) about \( Z \)-axis. The equation \( \phi = \text{ constant} \) describes the plane containing \( Z \)-axis and making an angle \( \phi \) to
the XZ-plane. The general differential displacement of particle $P$ in cylindrical co-
ordinates is given by

$$dr = d\rho \hat{e}_\rho + \rho d\phi \hat{e}_\phi + dze_z$$

where $\hat{e}_\rho$ and $\hat{e}_\phi$ are related to $(i, j, k)$ as given by
Eqs. 1.11a, b, and $e_z = k$. The unit vectors $(\hat{e}_\rho, \hat{e}_\phi, \hat{e}_z)$ constitute an orthogonal (right-handed) co-
ordinate system, i.e.,

$$\hat{e}_\rho \times \hat{e}_\phi = \hat{e}_z, \hat{e}_\phi \times \hat{e}_z = \hat{e}_\rho, \hat{e}_z \times \hat{e}_\rho = \hat{e}_\phi$$

and $$\hat{e}_\rho \cdot \hat{e}_\phi = \hat{e}_\rho \cdot \hat{e}_z = \hat{e}_\phi \cdot \hat{e}_z = 0$$

In situations where there is an axis of symme-
try, cylindrical co-ordinates become the right choice to describe the system. A typical example is the (planar) rotation of a particle about an axis.

In general, if a particle moves in a plane, we describe its motion in polar co-ordinates $(\rho, \phi)$, where $\rho$ gives the radial position of the particle. (We can call the plane as $Z = 0$, or XY-plane). The velocity and acceleration of the particle are then given by Eqs. 1.12 to 1.15, as discussed in last section. In particular, if the particle moves in a circle of radius $\rho$, we have (Eq. 1.14):

$$v = \rho \dot{\phi} \hat{e}_\phi = \vec{\omega} \times \vec{\rho}$$

where $\vec{\rho} = \rho \hat{e}_\rho$, and $\vec{\omega} = \dot{\phi} \hat{e}_z$. The vector $\vec{\omega}$ represents angular velocity whose magnitude is $\dot{\phi}$ and direction is about the axis of rotation.

The acceleration of the particle rotating about Z-axis can be rewritten as,

$$a = \frac{dv}{dt} = \frac{d\vec{\omega}}{dt} \times \vec{\rho} + \vec{\omega} \times \frac{d\vec{\rho}}{dt}$$

$$= \vec{\alpha} \times \vec{\rho} + \vec{\omega} \times (\vec{\omega} \times \vec{\rho})$$

where, $\vec{\omega} \times (\vec{\omega} \times \vec{\rho}) = \rho \dot{\phi}^2 \hat{e}_z \times (\hat{e}_z \times \hat{e}_\rho) = -\rho \dot{\phi}^2 \hat{e}_\rho$ is the centripetal acceleration, and

$$\vec{\alpha} \times \vec{\rho} = \rho \ddot{\phi} \left( \hat{e}_z \times \hat{e}_\rho \right) = \rho \ddot{\phi} \hat{e}_\phi$$

is the tangential acceleration.

Note that if the particle is rotating in the XY-plane, $\vec{\rho}$ corresponds to position vector of the particle. Hence, one usually denotes $\vec{\rho}$ by $r$ and hoping no confusion, we write

$$v = \vec{\omega} \times r$$

...(1.18a)

and

$$a = \vec{\omega} \times (\vec{\omega} \times r) + \vec{\alpha} \times r$$

...(1.18b)

We shall frequently need above relations when describing the motion in rotating frame, and/or rotating motion of an object.
1.2 NEWTON’S LAWS OF MOTION

In 1687, Isaac Newton articulated the laws upon which all classical mechanics is based. In present-day terminology these laws may be stated as follows:

**I Law:** An object continues in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by external forces acting on it.

**II Law:** The time rate of change of linear momentum of an object is proportional to the net external force acting on it and occurs in the direction of the resultant force. That is,

\[ F = k \frac{dp}{dt} = kma \]  

where \( F \) is resultant external force; \( m \) is inertial mass of the object, and \( p = m v \) is its linear momentum. \( k \) is constant of proportionality which is taken as 1 by defining appropriate units.

**III Law:** When two objects interact, the mutual forces of interaction on each object are equal in magnitude and opposite in direction. That is, ‘to every action there is an equal and opposite reaction.’

1.2.1 Comments on Newton’s Laws

**I Law and frames of reference:** Newton’s I law has two ingredients: first, that of force (or the absence of it) and second, that of measurement of rest or uniform motion of the object. Force is the expression for interaction between different objects. There is a priori, no way to know that there is no force acting on an object except by putting it so far away from all other objects that we would believe that it is not acted by an external force. Having such an object, Newton’s I law asserts that it would either remain at rest or continue to move with constant velocity. But, with respect to whom? Obviously, if the observer himself is sitting on an accelerated frame continuously changing his velocity, the object cannot always remain at rest or in uniform motion with respect to him. A reference frame with respect to which I law holds is called an inertial frame of reference. Thus, Newton’s first law defines the criterion for an inertial frame. The law says that there exists certain frames of reference (called inertial frames) with respect to which an object, (believed to be) not acted upon by any external force, stays either at rest or in uniform motion in a straight line.

Obviously, an inertial frame itself must be a frame moving with constant velocity, i.e., non-accelerating. The question whether a given frame of reference is inertial or not then becomes a matter of observation and experiment. For most of the observations made on the surface of Earth, the frame of reference attached to the surface of Earth behaves as an inertial frame to a very good approximation. More refined observations, which we shall discuss later, shows that it is not. However, unless otherwise stated, we shall refer to Earth’s frame as ‘the’ inertial frame. All other frames moving uniformly in straight line relative to Earth therefore also constitute inertial frames of reference.

Newton’s I law is an independent law and is not contained in his II law: In fact, both II and III laws hold only in an inertial frame suitably chosen on the basis of I law.

1.2.2 II Law

**Newton’s II law is not a definition of force:** The equation, \( F = ma \), is an equality
which says that the force acting on a body determines not the velocity but rate of change of velocity of the body. What is the nature and value of $F$ is left unanswered in II law.

Newton’s II law is valid for an object having constant inertial mass and moving with velocity small compared to that of light. If the velocity of the object is high (i.e., a significant fraction of velocity of light), we enter into the domain of special relativity. There we find inertial mass changes with velocity; direction of acceleration is generally not same as that of force; hence Newton’s law needs to be revised in relativistic dynamics.

1.2.3 III Law and Principle of Momentum Conservation

Newton’s III law is the only statement made by Newton about the nature of force. It says that no matter what is the exact form of a force (i.e., interaction) between two objects, it behaves in such a way that at any instant of time, if an object 1 exerts a force $F_{12}$ on object 2, then object 2 exerts an equal and opposite force $F_{21}$ on object 1. That is, at any instant

$$F_{12} + F_{21} = 0 \quad \text{(1.20)}$$

which implies

$$\frac{dp_1}{dt} + \frac{dp_2}{dt} = \frac{dP}{dt} = 0$$

or $P = p_1 + p_2$ is constant in time. \( \text{(1.21)} \)

Thus, when two objects interact, the sum of their momenta is constant. Newton’s III law, alternatively, is a statement of momentum conservation for a system of interacting objects (in absence of any external force).

In its original form, Newton’s III law has profound limitations. For example, it is not satisfied by two separate charges interacting through electromagnetic field. Propagation of electromagnetic interaction takes place at a finite speed $c$, speed of light. The transfer of momentum (i.e., travel of force field) from one charge to another involves a time interval equal to distance between the charges divided by $c$. Thus, at any instant, a change in momentum occurs in first charge without an equal and opposite change in momentum in the other charge. That is, instant by instant, we find

$$F_{12} \neq F_{21} \quad \text{or} \quad \Delta p_1 \neq \Delta p_2$$

Momentum conservation principle is eventually preserved by associating the instantaneous change in momentum of particles with that of the electromagnetic field which carries the interaction.

In fact, no interaction (whether electromagnetic, or gravitational, or any other) can propagate with speed greater than that of light. Hence, in general, momentum conservation principle for an isolated, interacting system of particles is valid only when we consider particles as well as fields that carry the interactions as the complete system. However, in this book, we shall not go into these complications and assume Newton’s III law to be valid in its strong form as given by Eq. 1.20.

1.3 SPACE-TIME SYMMETRY AND CONSERVATION LAWS

From our experience, we articulate the concepts of space and time to describe the motion of an object. It is from the same experience and observation, we ascribe certain fundamental properties to the concepts of space and time. One such property is homogeneity and the other is isotropy of space and time.
1.3.1 Homogeneity of Space

Space is homogeneous means one part (or point) of space is identical or equivalent to any other part (or point) of space. Any physical process will occur the same way (or identically) under identical conditions, no matter where (i.e., in whatever part of space) it occurs; a particular experiment, whether performed in Kanpur or New York, will yield same result. (Incidentally Kanpur and New York represent same inertial frame.)

Let us apply the basic property of homogeneity of space to an isolated system of two interacting particles: if both the particles of the system are displaced by same amount $\delta r$, there should be no change in the state of the system or in its internal motion. That is, the total work done by internal forces when the system is displaced by $\delta r$ must be zero:

$$F_{12} \cdot \delta r + F_{21} \cdot \delta r = 0$$

Since $\delta r$ is arbitrary, we get

$$F_{12} + F_{21} = 0$$

This is Newton’s III law which leads to law of momentum conservation for the isolated system of two particles. Thus, momentum conservation emerges from the fundamental property of homogeneity of space.

[The above argument can be extended to an isolated system of $n$-interacting particles. No work done by internal forces imply:

$$\delta r \cdot (F_{12} + F_{21}) + \delta r \cdot (F_{13} + F_{31}) + \ldots = 0$$

where the series includes all pairs of two particles out of $n$-particle system. That is, in short, we can write

$$\delta r \cdot \sum_{i \neq j} \left( F_{ij} + F_{ji} \right) = 0 \quad i, j = 1 \ldots n$$

Since $\delta r$ is arbitrary, we find

$$\sum_{i} \sum_{j} (F_{ij} + F_{ji}) = 0 \quad i, j = 1, \ldots, n; i \neq j$$

The comments made in Sec. 1.2.3 on Newton’s III law also apply to the above proof of momentum conservation (of particles) from homogeneity of space. If we consider the forces propagating with finite speed, homogeneity of space leads to conservation of momentum of complete system, viz, both particles and the carrier of force.

Since space is homogeneous, we can choose the origin of our co-ordinate system anywhere we wish. Shifting the origin means displacing the system; and it does not affect the processes.

1.3.2 Homogeneity of Time

Time is homogeneous means one instant of time (or duration) is identical to any other instant (or duration) of time. An experiment, whether performed today or tomorrow or a year later will yield same result under otherwise same conditions.

Let us discuss the effect of concept of homogeneity of time on Newton’s laws of motion. Homogeneity of time implies that the forces in nature should not depend explicitly on
time. [For example, the force $F$ of gravitation between two point masses $m_1$ and $m_2$ separated by distance $r$ remains the same at time $t$ (today) and $t'$ (tomorrow).]

That is, for all kinds of forces, we have

$$\frac{\partial F}{\partial t} = 0$$

Now if a force is conservative, we can define a potential energy of interaction $U$ such that the force is negative gradient of $U$, i.e., we have

$$F_x = -\frac{\partial U}{\partial x}, \quad F_y = -\frac{\partial U}{\partial y}, \quad F_z = -\frac{\partial U}{\partial z}$$

This means that if $F$ does not depend explicitly on $t$, then equivalently, potential energy $U$ also is not an explicit function of $t$, or,

$$U = U(x, y, z)$$

where $x, y, z$ refer to co-ordinates of the particle under consideration in a given inertial frame.

The explicit independence of $U$ on $t$ leads to conservation of mechanical energy of the particle. To show this, let us write the total energy of the particle as,

$$E = \frac{1}{2}mv^2 + U$$

Hence, we get

$$\frac{dE}{dt} = mv \frac{d\mathbf{v}}{dt} + \frac{dU}{dt}$$

$$= \mathbf{v} \cdot \mathbf{F} + \left( \frac{\partial U}{\partial x} \frac{dx}{dt} + \frac{\partial U}{\partial y} \frac{dy}{dt} + \frac{\partial U}{\partial z} \frac{dz}{dt} \right)$$

$$= \mathbf{v} \cdot \mathbf{F} + \left( -F_x v_x - F_y v_y - F_z v_z \right)$$

$$= 0$$

Or, $E$ remains constant in time. Thus the law of conservation of energy, in Newtonian mechanics, follows directly as a consequence of homogeneity of time.

Homogeneity of time implies that we can choose the zero of time at any instant for the observation of a physical process.

1.3.3 Isotropy of Space

Space is isotropic means one direction in space is identical or equivalent to any other direction. A particular experiment (i.e., physical process) will yield the same result whether our laboratory faces North or West. That is, an angular displacement of an isolated system does not change the internal state of the system, nor its internal motion.

When applied to an isolated system of two interacting particles, isotropy of space implies that the net work done by internal forces must be zero when the system is rotated by an angle $d\phi$. 

That is, $$\dot{d r}_1 \cdot \mathbf{F}_{21} + \dot{d r}_2 \cdot \mathbf{F}_{12} = 0$$

where \( \dot{d r}_1 \) and \( \dot{d r}_2 \) are displacement vectors of particles 1 and 2 respectively.

We know that \( \mathbf{v} = \bar{\omega} \times \mathbf{r} \); that implies,

$$\dot{d r}_1 = d \bar{\phi} \times \mathbf{r}_1, \text{ and } \dot{d r}_2 = d \bar{\phi} \times \mathbf{r}_2$$

Thus, we find

$$(d \bar{\phi} \times \mathbf{r}_1) \cdot \mathbf{F}_{21} + (d \bar{\phi} \times \mathbf{r}_2) \cdot \mathbf{F}_{12} = 0$$

or

$$d \bar{\phi} \cdot (\mathbf{r}_1 \times \mathbf{F}_{21}) + d \bar{\phi} \cdot (\mathbf{r}_2 \times \mathbf{F}_{12}) = 0$$

Since \( d \bar{\phi} \) is arbitrary, we get

$$\mathbf{r}_1 \times \mathbf{F}_{21} + \mathbf{r}_2 \times \mathbf{F}_{12} = 0$$

that is, the net torque produced on the system by internal forces is zero. This implies that the angular momentum of the system remains constant. The above analysis can be extended to an n-particles system.

Thus, conservation of angular momentum of an isolated system emerges as a consequence of a fundamental property of space, viz., its isotropy.

Isotropy of space implies we can orient our co-ordinate axes in space in any way we wish.

### 1.3.4 Isotropy of Time

Isotropy means equivalence of directions; isotropy of time means equivalence of directions of time. However, we usually refer to only one direction of time, the forward direction, from present to future. Nevertheless, we can conceive of a backward direction of time, i.e., from present to past. Therefore, the question whether time is isotropic or not means whether time is reversible or not? Operationally, it means to find out what happens if we change \( t \) by \(-t\)?

Newton’s laws of motion do not change if we replace \( t \) by \(-t\) in it:

$$m \frac{d^2 \mathbf{r}}{d(-t)^2} = m \frac{d^2 \mathbf{r}}{dt^2} = \mathbf{F}$$

Since \( \mathbf{F} \) does not explicitly contain \( t \), it remains unaffected. Thus, the motion described by Newton’s laws remains same when we replace \( t \) by \(-t\). What it means is the following: consider a physical process governed by Newton’s laws, as for example, motion of a particle thrown vertically upwards. It goes up, stops, and then falls back under gravity. Let us take a movie of the process as it proceeds forward in time. If we now view the movie in reverse, we look at the process occurring backward in time. The entire process looks normal: particle goes up, stops, and falls back exactly according to Newton’s laws. Thus, Newton’s laws are reversible in time.

Processes in systems comprising of large number of particles do not show time reversibility. As a typical example, consider a glass which falls from a table and shatters into small pieces on the floor. If we run a motion picture of the above process in reverse, we see the pieces gathering together on the floor into a whole glass, which then jumps onto the
table. This does not happen in nature. Thus, while each piece of glass follows Newton’s laws which are time reversible, the collective system breaks time reversibility. This is explained by II law of thermodynamics which prohibits a macroscopic system to move from less ordered to a more ordered state (law of increase of entropy).

Isotropy of time does not lead to any specific conservation principle in classical mechanics.

To sum up, in this section, we described the space-time symmetries and their consequences in understanding the nature of forces and conservation laws, within a given inertial frame. Now, we shall see how Newtonian space-time structure connects motion as observed from two different inertial frames.

1.4 PRINCIPLE OF RELATIVITY AND GALILEAN TRANSFORMATION

It was first observed by Galileo, in multitude of experiments, that mechanical phenomena occur identically in all inertial frames. (Later, this was experimentally verified for electromagnetic phenomena as well.) These observations of Galileo were stated by Newton in the following words: “The motions of bodies included in a given space are the same among themselves, whether that space is at rest or moves uniformly forward in a straight line.”

The above statement, known as Galilean principle of special relativity, means that all phenomena in nature should be related in same way in all inertial frames. In modern terminology, it says that the laws of nature must remain form-invariant as we go from one inertial frame to another.

The measurements of the motion of a particle P (i.e., its position, velocity, and acceleration at any instant) from two different inertial frames S and S' leads to two sets of values. The relationships between these two sets of measurements are called Galilean transformation equations.

One of the reference frames S is (arbitrarily) called stationary while the other S' is called moving. The two frames must be moving with a constant velocity \( V \) relative to each other. Otherwise, the two frames become same. Both the frames (or observers) S and S' may use Cartesian co-ordinate systems for measurements. Let us denote the co-ordinates of particle P in frame S by \( \mathbf{r} = (x, y, z) \) and time \( t \), and in frame S' by \( \mathbf{r}' = (x', y', z') \) and time \( t' \); \( t \) and \( t' \) denote the time read by clocks attached in frames S and S', at the instant when position co-ordinates of P are respectively \( \mathbf{r} \) and \( \mathbf{r}' \).

![Fig. 1.5 (a)](image1.png)

![Fig. 1.5 (b)](image2.png)
In particular, if we assume that origins $O$ and $O'$ of both the co-ordinate systems coincide at $t = t' = 0$, then at any later time $t = t'$, the origin $O'$ of co-ordinate system $S'$ will be at a distance $R = Vt$ from the origin $O$ of co-ordinate system $S$, where $V$ is the constant velocity with which $S'$ moves relative to $S$. Hence, if $r$ and $r'$ are position vectors of $P$ as observed from $S$ and $S'$ at that instant, then we have (see Fig. 1.5a).

$$r = R + r' = Vt + r'$$  \hspace{1cm} (1.21a)

or

$$r' = r - Vt$$  \hspace{1cm} (1.21b)

The above relation, along with $t = t'$, are known as Galilean transformations.

As a special case, if the frame $S'$ moves parallel relative to frame $S$ with $X'$-axis coinciding with $X$-axis, as shown in Fig. 1.5b, the Galilean transformations are given in components as following:

$$x' = x - Vt$$

$$y' = y$$  \hspace{1cm} (1.22)

$$z' = z$$

$$t' = t$$

Galilean transformations are not the transformation equations between two co-ordinate systems within a single inertial frame, like e.g., Eq. 1.5 which relate rectangular to spherical polar co-ordinates; these are relations between two different inertial frames.

### 1.5 INVARIANTS OF GALILEAN TRANSFORMATIONS

The numerical values of various physical quantities generally change under a co-ordinate transformation. Trivial example is the value of position co-ordinates $(x, y, z)$ of a particle $P$, which change to $(x', y', z')$ under the transformation. It means that the value of position co-ordinates itself has no objective reality, or in other words, it is not an objective property of the particle. It is simply the geometrical position of the particle relative to a particular co-ordinate system.

The physical quantities whose values do not change under a transformation are called the invariants of the transformation. Such physical quantities reflect the objective properties of the object (or phenomenon) under consideration in the sense that these properties remain independent of the choice of the co-ordinate system. In the following discussion, we look into the invariants of Galilean transformation.

#### 1.5.1 Invariance of Acceleration

Differentiating Eq. 1.21 with respect to time, we find that

$$\frac{dr}{dt} = \frac{dR}{dt} + \frac{d}{dt}r'$$

or

$$v = V + v'$$  \hspace{1cm} (1.23)

where $v' = \frac{d}{dt}r'$ is the velocity of $P$ relative to $S'$; $V$ is velocity of $S'$ relative to $S$, and $v$ is velocity of $P$ relative to $S$. Eq. 1.22 describes well known relation of velocity addition in Newtonian mechanics.
Differentiating once more, we get
\[ a = \frac{d\mathbf{v}}{dt} = \frac{d\mathbf{v}'}{dt'} = a' \]
\[ \text{...(1.24)} \]
because \( \mathbf{V} \) is constant. That is, acceleration of particle \( P \) is same when observed from two different inertial frames. Acceleration of the particle \( P \), therefore, remains invariant under Galilean transformation.

1.5.2 Invariance of Space and Time Intervals

Suppose there are two points \( P_1 \) and \( P_2 \) whose position vectors are \((\mathbf{r}_1, \mathbf{r}_2)\) and \((\mathbf{r}_1', \mathbf{r}_2')\) as observed from two frames \( S \) and \( S' \) respectively, at any given instant of time \( t = t' \). Then, according to Galilean transformation, we have
\[ \mathbf{r}_1' = \mathbf{r}_1 - \mathbf{V} t \]
and
\[ \mathbf{r}_2' = \mathbf{r}_2 - \mathbf{V} t \]
where \( \mathbf{V} \) is the constant velocity of \( S' \) relative to \( S \). Hence, we get
\[ \mathbf{r}_2' - \mathbf{r}_1' = \mathbf{r}_2 - \mathbf{r}_1 \]
or
\[ l' = |\mathbf{r}_2' - \mathbf{r}_1'| = |\mathbf{r}_2 - \mathbf{r}_1| = l \]
\[ \text{...(1.25)} \]

Eq. 1.25 means that,
\[ \sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \]

\( l' \) (or \( l \)) denotes the space interval, or spatial distance between the two points. It may also represent the length of a ‘rigid’ rod, for example, whose end points are \( P_1 \) and \( P_2 \). Thus, we find that space interval, or length of a rigid object in general, remains invariant under Galilean transformation.

Similarly, if two events occur at times \( t'_1 = t_1 \) and \( t'_2 = t_2 \), the time interval between these events as observed from two inertial frames is same, that is,
\[ t'_2 - t'_1 = t_2 - t_1 \]

In other words, time interval is also an invariant of Galilean transformation.

1.5.3 Galilean Invariance of Newton's Laws and Nature of Forces

Principle of (special) relativity demands that all laws of nature should be valid in all inertial frames. This is obviously satisfied by Newton’s I law: if the velocity of a particle is constant \( (\mathbf{v}) \) in frame \( S \), it remains constant \( (\mathbf{v}') \) in frame \( S' \) as well,
\[ \mathbf{v}' = \mathbf{v} - \mathbf{V} \]
\[ \text{...(1.26)} \]
because relative velocity \( (\mathbf{V}) \) between the two inertial frames is a constant.

Let us now look into Newton’s II law. Suppose a particle of mass \( m \) moves with the acceleration \( \mathbf{a} \) due to an external force \( \mathbf{F} \), as observed from frame \( S \).
Then we have,

$$\mathbf{F} = m \mathbf{a} \quad \text{ ...(1.27)}$$

If the II law has to be form-invariant under Galilean transformation, then in frame $S'$, we should get the following relation:

$$\mathbf{F}' = m \mathbf{a}' \quad \text{ ...(1.28)}$$

where $\mathbf{F}'$ and $\mathbf{a}'$ are the force and acceleration as observed from $S'$. It is assumed that inertial mass $m$, being a scalar, is a frame independent quantity.

We know that acceleration is an invariant of Galilean transformation, i.e., $\mathbf{a}' = \mathbf{a}$. Hence, if both the Eqs. 1.27 and 1.28 have to hold, then we must have,

$$\mathbf{F}' = \mathbf{F} \quad \text{ ...(1.29)}$$

The above condition imposes a severe condition on the nature of forces; that is, forces appearing in nature must be invariants of Galilean transformation if Newton's II law has to be valid in all inertial frames.

How do we know that $\mathbf{F}' = \mathbf{F}$? What kind of function $\mathbf{F}$ (or $\mathbf{F}'$) should be in order to satisfy above equality? It turns out that all the forces we encounter in Newtonian dynamics depend only on the relative position or relative velocity of two interacting bodies (and not on their absolute position or velocities in space). That is, as observed from frame $S$, a force exerted by particle 2 on particle 1 is found to have the following functional dependence,

$$\mathbf{F}_{21} = f(r_2 - r_1, v_2 - v_1) \quad \text{ ...(1.30)}$$

Hence, as seen from $S'$, the same force would transform to,

$$\mathbf{F}'_{21} = f(r'_2 - r'_1, v'_2 - v'_1)$$

Since, under Galilean transformation, we have

$$r'_2 - r'_1 = r_2 - r_1$$

and

$$v'_2 - v'_1 = v_2 - v_1 \quad \text{ ...(1.31)}$$

we find

$$\mathbf{F}'_{21} = \mathbf{F}_{21}$$

That is, the forces found in nature happen to be invariants of Galilean transformation, thus making Newton's II law valid in all inertial frames.

Further, if forces in nature have a functional dependence given by Eq. 1.30, then we find

$$\mathbf{F}_{12} = f(r_2 - r_1, v_2 - v_1) = -f(r_1 - r_2, v_1 - v_2) = -\mathbf{F}_{21} \quad \text{ ...(1.32)}$$

which is the statement of Newton's III law. Thus, the III law implicitly ensures that both II and III laws remain valid in all inertial frames. (Note that $\mathbf{F}'_{12} = \mathbf{F}_{12} = -\mathbf{F}_{21} = -\mathbf{F}'_{21}$.)

All the Newton's laws of motion therefore remain form-invariant under Galilean transformation in a self-consistent manner.

**Example 1.2:** Show that the conservation of total momentum of an isolated system of two colliding particles in the process of collision follows from the principle of Galilean invariance and the law of conservation of energy.
Solution: To show this, let us consider the collision of two particles 1 and 2 as observed from an inertial frame $S$. Suppose initially 1 and 2, moving with velocities $u_1$ and $u_2$, are so far apart that they do not possess any potential energy of interaction. The total energy of the system of both particles is just their kinetic energy,

$$\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2$$

After they collide and separate far apart, let their velocities be $v_1$ and $v_2$. According to law of conservation of energy in frame $S$, we get

$$\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \Delta E$$

where $\Delta E$ is the change (loss or gain) in internal energy of the particles during collision. If $\Delta E = 0$, collision is elastic; otherwise inelastic.

Now let us view the process from another inertial frame $S'$ moving with velocity $V$ relative to $S$. According to Galilean transformation, the velocities of particles as observed from $S'$ are,

$$u'_1 = u_1 - V,$$  
$$u'_2 = u_2 - V$$  
(before)  

$$v'_1 = v_1 - V,$$  
$$v'_2 = v_2 - V$$  
(after)  

Now we introduce the idea that principle of conservation of energy is frame independent. In fact, conservation of mechanical energy being a consequence of homogeneity of time, is fundamentally an invariant principle in Newtonian mechanics. As shown by experiments, we further assume that internal energy change $\Delta E$ also remains invariant under Galilean transformations.

Hence, conservation of energy in collision process as observed in $S'$, gives

$$\frac{1}{2} m_1 (u'_1)^2 + \frac{1}{2} m_2 (u'_2)^2 = \frac{1}{2} m_1 (v'_1)^2 + \frac{1}{2} m_2 (v'_2)^2 + \Delta E$$

where $V$ is the velocity of frame $S'$ relative to frame $S$. Using Eq. 1.34, we find

$$(u'_1)^2 = u_1^2 + V^2 - 2u_1 \cdot V$$

etc. Substituting for $(u'_1)^2, (u'_2)^2, (v'_1)^2, (v'_2)^2$ in Eq. 1.35, we get

$$\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 - (m_1 u_1 + m_2 u_2) \cdot V = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 - (m_1 v_1 + m_2 v_2) \cdot V + \Delta E$$

Comparing above with Eq. 1.33, we get

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

which is the law of conservation of momentum in collision process. Thus, the principle of conservation of energy and the concept of Galilean invariance shows that momentum remains conserved during collision, whether elastic or inelastic.

Of course, momentum conservation in collision process follows rather straightforward from Newton’s III law. If we consider the two colliding particles as an isolated combined system on which no external force acts, then the momentum of the system must remain
conserved, no matter whether they collide or do something else. Moreover, momentum conservation being a consequence of homogeneity of space must also be a fundamentally Galilean invariant principle. That is, in frame $S'$, we have

$$m_1 \mathbf{u}_1 + m_2 \mathbf{u}_2 = m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2$$

### 1.6 SOME BASIC FORCES IN NATURE

A force expresses the interaction between two different objects. We are familiar with a large variety of forces such as gravitational force, electric and magnetic force, and intermolecular forces giving rise to dry friction, viscosity, surface tension, tension in the string, or elastic force in a spring, etc. We are also familiar with nuclear forces responsible for binding of protons and neutrons together in a nucleus, or the splitting of a neutron into a proton, electron and anti-neutrino in beta-decay. Among this large diversity of terms, it turns out that there are only four types of basic forces (known at present) in nature, viz.,

1. Gravitational force
2. Electromagnetic force
3. Strong nuclear force
4. Weak force.

We shall now briefly outline the basic characteristic of these forces.

1. **Gravitational force:** The gravitational interaction is a universal phenomenon which acts between all forms of matter and energy. The quantitative nature of gravitational force was first given by Newton; according to Newton's law of gravitation, between any two material particles there exists an attractive force of gravitation, given by

   $$F_{12} = G \frac{m_1 m_2}{r_{12}^2}$$

   where $m_1$ and $m_2$ are the gravitational masses and $r_{12}$ is the relative distance between the (interacting) particles. The value of gravitational constant $G$ is about $6.7 \times 10^{-11}$ Nm$^2$/kg$^2$.

   The gravitational interaction plays significant role only when masses involved are large. It becomes the decisive force when we study motions of planets, stars, or other such cosmic objects. We shall study more about the nature of gravitational interaction in Ch 7. The gravitational interaction between earth and other relatively smaller objects near the surface of earth produces an almost uniform acceleration due to gravity $g = 9.8$ m/s$^2$ (approximately) in all these objects. Hence, in considering the motion of any such prototype object, e.g., a block or a sphere of mass $m$ on the surface of earth, it is implicit that gravity force $mg$ is acting on the object. On the other hand, if we are considering the motion of microscopic particles like electrons etc., the gravity force is completely neglected because of the tiny masses of these particles (as we discuss in Ch. 3; also see Problem 5, set A, Ch. 1).

2. **Electro-magnetic force:** The second kind of fundamental force is the electro-magnetic force that acts between charged particles. The first law of electro-magnetism is the Coulomb's law of electric force between two charges $q_1$ and $q_2$ fixed (i.e., kept at rest) at relative distance $r_{12}$. The Coulomb's law states that the electric force between these static charges is given by

   $$F_{12} = \frac{1}{4\pi \varepsilon_0} \frac{q_1 q_2}{r_{12}^2}$$